The three-dimensional transport equation with applications to energy deposition and reflection

1982 J. Phys. A: Math. Gen. 15 965

(http://iopscience.iop.org/0305-4470/15/3/032)

J. Phys. A: Math. Gen. 15 (1982) 965–983. Printed in Great Britain

The three-dimensional transport equation with applications to energy deposition and reflection

M M R Williams

Nuclear Engineering Department, Queen Mary College, University of London, Mile End Road, London E1 4NS, UK

Received 9 July 1981

Abstract. A detailed investigation of the energy deposition in, and surface reflection from, an infinite half-space has been made. Two types of source are considered: the first is a line source embedded in the medium perpendicular to the surface and the second is an incident pencil, of arbitrary direction, incident at a point. The resulting problem involves three dimensions in space and therefore requires description by a transport equation in the appropriate coordinates.

The physical problem considered is that of a beam of incident ions or a line ion source in the medium. Only the fate of the foreign incident ions is considered and no attempt is made to follow the recoil atoms generated. Progress is made in the analytical solution of the problem by assuming an energy-independent mean free path and the transport approximation for the scattering kernel. The Wiener–Hopf method is used together with Fourier transforms for transverse directions. Considerable success has been achieved in obtaining exact solutions for some special limiting cases, and the numerical results which emerge are tabulated.

1. Introduction

The distribution of radiation damage and energy deposition in materials due to spatially non-uniform sources is of considerable interest. A simple, yet important, example is the case of a pencil of radiation incident on a surface. As the direction of the pencil is altered the distribution of radiation within the medium and the scattered distribution vary. In astrophysics this is known as the searchlight problem (Rybicki 1971). Similar problems also arise in neutron transport, where Elliott (1952, 1955) has studied the distribution of neutrons arising from a point isotropic source on the surface of a half-space. Extensions of this type of problem in the field of radiation damage have been made by Khalafi and Williams (1980), who consider the energy deposited in an infinite medium by an anisotropic point source.

The purpose of the present paper is to examine the energy deposition in a half-space of moderating material arising from a pencil source and also that due to a line source embedded in the medium perpendicular to the surface. Because of the difficulty of dealing with such problems both numerically and analytically, we shall make a number of assumptions, all of which are physically reasonable, which will enable an analytical solution to be obtained. The Wiener–Hopf method is used and enables the exact boundary conditions of the problem to be included. This has the advantage of eliminating the errors inherent in the conventional infinite-medium approximation.

0305-4470/82/030965+19$02.00

Explicit expressions are obtained for the emerging angular distribution, and for the energy deposited, as a function of position in the three orthogonal directions. Various difficulties encountered in the Fourier inversions are discussed.

2. Three-dimensional transport equation

The structure of the three-dimensional transport equation describing the motion of foreign particles in a host medium can be found in several references (Davison 1957, Case and Zweifel 1967). However, for consistency, we refer the reader to Williams (1979a, b). Then, if $\Phi(E, \boldsymbol{\Omega}, x, y, z)$ is the collision density of particles of energy E travelling in the direction denoted by the unit vector $\boldsymbol{\Omega}$ at position x, y, z, Φ is given by

$$\begin{aligned}(\lambda_{\rm tr}(E)\boldsymbol{\Omega}\cdot\nabla+1)\Phi(E,\boldsymbol{\Omega},x,y,z)\\ =\frac{1}{2\pi(1-\alpha)}\int_E^{E/\alpha}\frac{\mathrm{d}E'}{E'}f(\theta_{\rm c}(E/E'))\int \mathrm{d}\boldsymbol{\Omega}'\,\delta(\mu_0^*-g(E/E'))\bigg\{\Phi(E',\boldsymbol{\Omega}',x,y,z)\\ +\left[\frac{E}{E'}\frac{1+\alpha}{1-\alpha}-\frac{2}{1-\alpha}\left(\frac{E}{E'}\right)^2\right]\Phi(E,\boldsymbol{\Omega},x,y,z)\bigg\}+S(E,\boldsymbol{\Omega},x,y,z).\end{aligned}\tag{1}$$

In this equation, we have assumed that the cross section in the centre of mass can be written as a separable function of E and $\theta_{\rm c}$, namely

$$\Sigma(E,\theta_{\rm c})=\Sigma_{\rm tr}(E)f(\theta_{\rm c})/4\pi \tag{2}$$

where $\Sigma_{\rm tr}=1/\lambda_{\rm tr}$ is the transport cross section referred to CM coordinates. $\theta_{\rm c}(E/E')$ is the scattering angle in the CM system and is a known function of E/E'. Similarly, $g(E/E')$ is the scattering angle in the laboratory system. $S(E, \boldsymbol{\Omega}, x, y, z)$ is a source term to be specified. In addition, there will be a boundary condition on any surface. For the sake of simplicity, we shall consider a half-space $z>0$ and hence we require a boundary condition of the form

$$\phi(E,\boldsymbol{\Omega},x,y,0)=F(E,\boldsymbol{\Omega},x,y)\qquad \boldsymbol{n}\cdot\boldsymbol{\Omega}>0 \tag{3}$$

where $\boldsymbol{n}$ is the unit normal to the surface.

It is frequently more convenient to work in terms of the lethargy variable $u=\ln(E_0/E)$ where E_0 is a convenient reference energy. In that case, the equation for Φ now becomes

$$\begin{aligned}(\lambda_{\rm tr}(u)\boldsymbol{\Omega}\cdot\nabla+1)\Phi(u,\boldsymbol{\Omega},x,y,z)\\ =\frac{1}{2\pi(1-\alpha)}\int_{u-q}^{u}\mathrm{d}u'\,\mathrm{e}^{u'-u}f(\theta(u-u'))\int\mathrm{d}\boldsymbol{\Omega}'\,\delta(\mu_0^*-g(u-u'))\\ \times\left[\Phi(u',\boldsymbol{\Omega}',x,y,z)+\left(\frac{1+\alpha}{1-\alpha}-\frac{2}{1-\alpha}\mathrm{e}^{u'-u}\right)\Phi(u,\boldsymbol{\Omega},x,y,z)\right]\\ +S(u,\boldsymbol{\Omega},x,y,z)\end{aligned}\tag{4}$$

where $q=\ln(1/\alpha)$.

In this work, two problems will be considered. They are
(1) the searchlight problem, in which

$$S(u, \boldsymbol{\Omega}, x, y, z) = 0 \tag{5}$$

$$\phi(u, \boldsymbol{\Omega}, x, y, 0) = \delta(u)\frac{\delta(\mu-\mu_0)}{\mu_0}\delta(\psi-\psi_0)\delta(x)\delta(y) \qquad \boldsymbol{n}\cdot\boldsymbol{\Omega}>0 \tag{6}$$

where $\boldsymbol{\Omega} = \boldsymbol{\Omega}(\mu, \psi)$ and
(2) the line source, in which

$$S(u, \boldsymbol{\Omega}, x, y, z) = S_0\delta(u)\delta(x)\delta(y)/4\pi \tag{7}$$

$$\phi(u, \boldsymbol{\Omega}, x, y, 0) = 0 \qquad \boldsymbol{n}\cdot\boldsymbol{\Omega}>0. \tag{8}$$

To proceed with the solution of equation (4), we make the assumption that the cross section Σ_{tr} is independent of energy, and scale all lengths in terms of it. Then the transport approximation to the scattering kernel is applied, in which the delta function is approximated as

$$\delta(\mu_0^* - g) \simeq \tfrac{1}{2}(1-g) + g\delta(\mu_0^* - 1). \tag{9}$$

This approximation has been studied in depth and found to give excellent results over a wide range of mass numbers (Williams 1978).

Using these approximations in equation (4) and taking Laplace transforms in lethargy, equation (4) reduces to

$$(\boldsymbol{\Omega}\cdot\nabla+1)\bar{\Phi}(s, \boldsymbol{\Omega}, x, y, z) = \frac{c(s)}{4\pi}\int \mathrm{d}\boldsymbol{\Omega}'\,\bar{\Phi}(s, \boldsymbol{\Omega}', x, y, z) + \bar{S}(s, \boldsymbol{\Omega}, x, y, z) \tag{10}$$

where x, y and z are now scaled by $\lambda_{tr}/(1-\Lambda(s))$ with

$$\Lambda(s) = \frac{1}{1-\alpha}\int_0^q \mathrm{d}u\, f(u)\left(g(u)\,\mathrm{e}^{-(s+1)u} + \frac{1+\alpha}{1-\alpha}\,\mathrm{e}^{-u} - \frac{2}{1-\alpha}\,\mathrm{e}^{-2u}\right) \tag{11}$$

$$c(s) = \hat{c}(s)/(1-\Lambda(s)) \tag{12}$$

$$\hat{c}(s) = \frac{1}{1-\alpha}\int_0^q \mathrm{d}u\,\mathrm{e}^{-(s+1)u}f(u)(1-g(u)) \tag{13}$$

$$\bar{\Phi}(s, \boldsymbol{\Omega}, x, y. z) = \int_0^\infty \mathrm{d}u\,\mathrm{e}^{-su}\Phi(u, \boldsymbol{\Omega}, x, y, z). \tag{14}$$

Equation (10) is now in the 'one-speed' form and is amenable to solution by a variety of techniques. It is this equation that Khalafi and Williams (1980), Elliott (1952, 1955) and Rybicki (1971) have solved for various boundary conditions.

3. Solution of the three-dimensional transport equation

We introduce the two-dimensional Fourier transform

$$\bar{\bar{\Phi}}(s, \boldsymbol{\Omega}, k_1, k_2, z) = \int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y\,\exp(-\mathrm{i}k_1x - \mathrm{i}k_2y)\bar{\Phi}(s, \boldsymbol{\Omega}, x, y, z) \tag{15}$$

whence equation (10) becomes

$$\left(\mathrm{i}\Omega_x k_1 + \mathrm{i}\Omega_y k_2 + \Omega_z \frac{\partial}{\partial z} + 1\right)\bar{\bar{\Phi}}(s, \boldsymbol{\Omega}, k_1, k_2, z)$$

$$= \frac{c}{4\pi} \int \mathrm{d}\boldsymbol{\Omega}' \, \bar{\bar{\Phi}}(s, \boldsymbol{\Omega}', k_1, k_2, z) + \bar{\bar{S}}(s, \boldsymbol{\Omega}, k_1, k_2, z). \tag{16}$$

In the searchlight problem $\bar{\bar{S}} = 0$ and

$$\bar{\bar{\Phi}}(s, \boldsymbol{\Omega}, k_1, k_2, 0) = \frac{\delta(\mu - \mu_0)}{\mu_0} \delta(\psi - \psi_0) \qquad \boldsymbol{n} \cdot \boldsymbol{\Omega} > 0 \tag{17}$$

whereas for the line source $\bar{\bar{\Phi}}(s, \boldsymbol{\Omega}, x, y, 0) = 0$ for $\boldsymbol{n} \cdot \boldsymbol{\Omega} > 0$ and

$$\bar{\bar{S}}(s, \boldsymbol{\Omega}, k_1, k_2, 0) = S_0/4\pi. \tag{18}$$

3.1. *The line source problem*

Noting that $\Omega_x = (1-\mu^2)^{1/2} \cos \psi$, $\Omega_y = (1-\mu^2)^{1/2} \sin \psi$ and $\Omega_z = \mu$, equation (16) can be written as

$$\left(\mu \frac{\partial}{\partial z} + 1 + \mathrm{i}(1-\mu^2)^{1/2}(k_1 \cos \psi + k_2 \sin \psi)\right)\bar{\bar{\Phi}}(s, \boldsymbol{\Omega}, k_1, k_2, z)$$

$$= \frac{c}{4\pi} \int \mathrm{d}\boldsymbol{\Omega}' \, \bar{\bar{\Phi}}(s, \boldsymbol{\Omega}', k_1, k_2, z) + \frac{S_0}{4\pi}. \tag{19}$$

Defining the total collision density $\bar{\bar{\Phi}}_0$ as

$$\bar{\bar{\Phi}}_0(s, k_1, k_2, z) = \frac{1}{2\pi} \int \mathrm{d}\boldsymbol{\Omega} \, \bar{\bar{\Phi}}(s, \boldsymbol{\Omega}, k_1, k_2, z) \tag{20}$$

equation (19) may readily be converted to the following integral equation for $\bar{\bar{\Phi}}_0$:

$$\bar{\bar{\Phi}}_0(s, k_1, k_2, z) = \tfrac{1}{2} \int_0^\infty \mathrm{d}z' \, K(|z-z'|; k)(c\bar{\bar{\Phi}}_0(s, k_1, k_2, z') + S_0) \tag{21}$$

where $k = (k_1^2 + k_2^2)^{1/2}$ and

$$K(|z|; k) = \int_0^1 \frac{\mathrm{d}\mu}{\mu} \frac{1}{(1+k^2\mu^2)^{1/2}} \exp\left(-\frac{|z|}{\mu}(1+k^2\mu^2)^{1/2}\right). \tag{22}$$

The angular distribution at the surface can be written as

$$\bar{\bar{\Phi}}(s, \boldsymbol{\Omega}, k_1, k_2, 0) = -\frac{c}{4\pi\mu} \int_0^\infty \mathrm{d}z' \, \bar{\bar{\Phi}}_0(s, k_1, k_2, z') \exp[z'(1+\mathrm{i}f)/\mu] \tag{23}$$

for $\mu < 0$ and all ψ where $f = (1-\mu^2)^{1/2}(k_1 \cos \psi + k_2 \sin \psi)$. Thus, if $\bar{\bar{\Phi}}_0$ can be obtained, the emergent distribution is available.

The basic problem now is to solve equation (21). This can be done by direct application of Fourier techniques; however, we prefer to adopt an alternative method based upon a device introduced by the author some years ago (Williams 1968). In this method, we introduce a fictitious function $\Psi(\eta, z)$ such that

$$\bar{\Phi}_0(\ldots z)=\int_{-1}^{1} \mathrm{d}\eta\ \Psi(\eta, z) \equiv \Psi_0(z). \tag{24}$$

The variable η has no physical significance, but it is readily shown that if $\Psi(\eta, z)$ satisfies the equation

$$\left(\eta Q \frac{\partial}{\partial z}+Q^2\right)\Psi(\eta, z)=\tfrac{1}{2}c\Psi_0(z)+\tfrac{1}{2}S_0 \tag{25}$$

where $Q=(1+k^2\eta^2)^{1/2}$ subject to

$$\Psi(\eta, 0)=0 \qquad \eta>0 \tag{26}$$

then $\Psi_0(z)$ satisfies equation (21). Application of the Wiener–Hopf method (Williams 1971) to equation (25) is rather easier than to the original equation; thus we introduce the Laplace transform

$$\bar{\Psi}(\eta, p)=\int_0^{\infty} \mathrm{d}z\ \mathrm{e}^{-pz}\Psi(\eta, z). \tag{27}$$

Note that in terms of $\bar{\Psi}(\eta, p)$ we can write the emergent distribution as

$$\bar{\Phi}(s, -\mu, \psi, k_1, k_2, 0)=\frac{c}{4\pi\mu}\bar{\Psi}_0\left(\frac{1+\mathrm{i}f}{\mu}\right). \tag{28}$$

We now develop the solution of equation (25) through the Wiener–Hopf technique. After applying the transform (27) to equation (25), dividing by $(\eta p+Q)Q$ and integrating over all η, we obtain

$$R(p)V(p)=S_0+cpg(p) \tag{29}$$

where

$$g(p)=\int_{-1}^{0} \frac{\eta\Psi(\eta, 0)\,\mathrm{d}\eta}{\eta p+Q} \tag{30}$$

$$R(p)=cp\bar{\Psi}_0(p)+S_0 \tag{31}$$

$$V(p)=1-\frac{c}{2(p^2-k^2)^{1/2}}\ln\left(\frac{1+(p^2-k^2)^{1/2}}{1-(p^2-k^2)^{1/2}}\right). \tag{32}$$

Now the basic idea behind the Wiener–Hopf technique is to rearrange equation (29) so that each side is analytic in overlapping half-planes. As the equation stands, we observe that $R(p)$ is analytic in $\mathrm{Re}(p)>0$, $g(p)$ is analytic in $\mathrm{Re}(p)<(1+k^2)^{1/2}$ and $V(p)$ is analytic in $-(1+k^2)^{1/2}<\mathrm{Re}(p)<(1+k^2)^{1/2}$. It is therefore necessary to decompose $V(p)$ as follows. Define

$$\tau(p)=\frac{p^2-1-k^2}{p^2-\nu^2-k^2}V(p) \tag{33}$$

where $\pm\nu$ are the roots of

$$1=\frac{c}{2\nu}\ln\left(\frac{1+\nu}{1-\nu}\right). \tag{34}$$

$\tau(p)$ is now free from zeros in the strip $-(1+k^2)^{1/2}<\mathrm{Re}(p)<(1+k^2)^{1/2}$, and tends to unity as $|p|\to\infty$. These are the conditions necessary for the decomposition

$$\tau(p)=\tau_+(p)/\tau_-(p)$$

where

$$\ln \tau_\pm(p)=\frac{1}{2\pi \mathrm{i}}\int_{\pm\beta-\mathrm{i}\infty}^{\pm\beta+\mathrm{i}\infty}\frac{\mathrm{d}u}{u-p}\ln \tau(u). \tag{35}$$

$\tau_\pm(p)$ are readily seen to be analytic in $\mathrm{Re}(p)<\beta$ and $\mathrm{Re}(p)>-\beta$, respectively, where $\beta<(1+k^2)^{1/2}$.

Now inserting τ_+ and τ_- into equation (29) and rearranging we get

$$\frac{(S_0+cpg(p))}{\tau_+(p)}\frac{p-(1+k^2)^{1/2}}{p-(\nu^2+k^2)^{1/2}}=\frac{R(p)}{\tau_-(p)}\frac{p+(\nu^2+k^2)^{1/2}}{p+(1+k^2)^{1/2}}. \tag{36}$$

The right-hand side is analytic in $\mathrm{Re}(p)>0$ and the left-hand side in $\mathrm{Re}(p)<-\beta$. Thus each side is the analytic continuation of the other. The behaviour as $|p|\to\infty$ gives the function by Liouville's theorem. Since each side tends to a constant C_0 as $|p|\to\infty$ we can write

$$\frac{(S_0+cpg(p))}{\tau_+(p)}\frac{p-(1+k^2)^{1/2}}{p-(\nu^2+k^2)^{1/2}}=C_0 \tag{37}$$

and

$$\frac{R(p)}{\tau_-(p)}\frac{p+(\nu^2+k^2)^{1/2}}{p+(1+k^2)^{1/2}}=C_0. \tag{38}$$

Hence from the definition of $R(p)$

$$cp\bar{\Psi}_0(p)+S_0=C_0\tau_-(p)\frac{p+(1+k^2)^{1/2}}{p+(\nu^2+k^2)^{1/2}}. \tag{39}$$

To obtain the value of C_0 we note that from the Tauberian theorem

$$\lim_{p\to 0} p\bar{\Psi}_0(p)=\Psi_0(\infty). \tag{40}$$

But clearly the value of $\Psi_0(\infty)$ is given by equation (25) with no spatial variation, namely

$$Q^2\Psi(\eta,\infty)=\tfrac{1}{2}c\Psi_0(\infty)+\tfrac{1}{2}S_0. \tag{41}$$

Hence

$$\Psi_0(\infty)=\frac{(S_0/k)\tan^{-1}k}{1-(c/k)\tan^{-1}k}. \tag{42}$$

From (39)

$$c\Psi_0(\infty)+S_0=C_0; \tag{43}$$

hence

$$C_0=\frac{S_0}{1-(c/k)\tan^{-1}k}. \tag{44}$$

Thus we have an explicit expression for $\bar{\Psi}_0(p)$ which we write as

$$\bar{\Psi}_0(p) = S_0[H(\infty)H(1/p) - 1]/cp \tag{45}$$

where

$$H(1/p) = \frac{p + (1+k^2)^{1/2}}{p + (\nu^2 + k^2)^{1/2}} \tau_-(p). \tag{46}$$

$H(Z)$ has been defined in this way because it reduces to the conventional Chandrasekhar H function for $k = 0$ (Chandrasekhar 1960). It is possible to obtain some relationships for the function $H(Z)$ which simplify the analysis. These are

$$H(\infty) = [1 - (c/k)\tan^{-1} k]^{-1/2} \tag{47}$$

$$\frac{1}{H(Z)} - \frac{1}{H(1/p)} = \tfrac{1}{2}c(1 - Zp)\int_0^1 \frac{\mathrm{d}\mu\, \mu H(\mu/Q)}{(Q + \mu p)(\mu + ZQ)Q} \tag{48}$$

$$\frac{1}{H(1/p)} = \tfrac{1}{2}c\int_0^1 \frac{\mathrm{d}\mu}{Q} H\left(\frac{\mu}{Q}\right)\left(\frac{1}{Q - \mu(\nu^2 + k^2)^{1/2}} - \frac{1}{Q + \mu p}\right). \tag{49}$$

By allowing $p \to 0$ or ∞, a number of other useful relations are obtained.

From equation (28) we see that

$$\Phi(s, -\mu, \psi, k_1, k_2, 0) = \frac{cS_0}{4\pi(1 + \mathrm{i}f)}\left(H(\infty)H\left(\frac{\mu}{1 + \mathrm{i}f}\right) - 1\right). \tag{50}$$

A further quantity of interest is the collision density at the surface of the half-space. We note that

$$\begin{aligned}\Psi_0(0) &= \lim_{p\to\infty} p\bar{\Psi}_0(p) \\ &= S_0(H(\infty) - 1)/c \\ &= \frac{S_0}{c}\left(\frac{1}{[1 - (c/k)\tan^{-1} k]^{1/2}} - 1\right).\end{aligned} \tag{51}$$

To regain the spatial dependence in the x–y plane and the z direction it is necessary to invert the various transforms. Similarly, to regain the energy dependence it is necessary to invert in the s plane. We note, however, that to calculate energy deposition (Williams 1979b), $W(x, y, z)$, we must obtain

$$\begin{aligned}W(x, y, z) &= \tfrac{1}{2}E_0(1 - \alpha)\int_0^\infty \mathrm{d}u\, \mathrm{e}^{-u}\Phi_0(u, x, y, z) \\ &= \tfrac{1}{2}(1 - \alpha)E_0\bar{\Phi}_0(1, x, y, z).\end{aligned} \tag{52}$$

Thus it is not necessary to invert in the s plane to obtain W. This does not apply, however, to the emergent distribution and since such an inversion involves considerable complexity (Williams 1979) we shall defer any further discussion of such distributions.

As an example of the spatial distribution in the x–y plane we examine the inversion of equation (51). Thus

$$W(x, y, 0)=\frac{E_0S_0(1-\alpha)}{2c}\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{-\infty}^{\infty}\mathrm{d}k_2\exp(\mathrm{i}k_1x+\mathrm{i}k_2y)(H(\infty)-1). \tag{53}$$

Because $H(\infty)$ depends only on k, the integrals over k_1 and k_2 can be reduced to

$$W(x, y, 0)=\frac{E_0S_0(1-\alpha)}{2c}\frac{1}{2\pi}\int_0^{\infty}\mathrm{d}k\, kJ_0(kR)\{[1-(c/k)\tan^{-1}k]^{-1/2}-1\} \tag{54}$$

where $R^2=x^2+y^2$.

Numerical evaluation of this integral is not easy because of the oscillatory nature of the Bessel function and we therefore seek a better behaved form. Note that the quantity in curly brackets has the Fourier integral

$$F(y)=\int_{-\infty}^{\infty}\mathrm{d}k\,\mathrm{e}^{\mathrm{i}ky}\{[1-(c/k)\tan^{-1}k]^{-1/2}-1\}. \tag{55}$$

Now the integrand has branch points at $k=\pm\mathrm{i}$ and at $k=\pm\mathrm{i}\nu$ where ν is the root of equation (34). If in the complex k plane the branch points $\mathrm{i}\nu$ and i are joined by a cut and $-\mathrm{i}\nu$ and $-\mathrm{i}$ are similarly joined we may deform the contour along the real axis to obtain

$$F(y)=2\int_{\nu}^{1}\frac{\mathrm{d}t\,\mathrm{e}^{-t|y|}}{[(c/2t)\ln[(1+t)/(1-t)]-1]^{1/2}}. \tag{56}$$

We may then regain the integrand in equation (55) by using the inverse transform, namely

$$\{[1-(c/k)\tan^{-1}k]^{-1/2}-1\}=\frac{2}{\pi}\int_{\nu}^{1}\frac{\mathrm{d}t\,t}{[(c/2t)\ln[(1+t)/(1-t)]-1]^{1/2}}\frac{1}{t^2+k^2}. \tag{57}$$

Inserting equation (57) back into equation (54) and using the relation

$$\int_0^{\infty}\frac{\mathrm{d}k\,kJ_0(kR)}{t^2+k^2}=K_0(tR) \tag{58}$$

we find

$$W(x, y, 0)=\frac{E_0S_0(1-\alpha)}{2\pi^2c}\int_{\nu}^{1}\frac{\mathrm{d}t\,tK_0(tR)}{[(c/2t)\ln[(1+t)/(1-t)]-1]^{1/2}}. \tag{59}$$

Recalling the scaling factor introduced in equation (10) we note that R should be replaced by $R(1-\Lambda(1))$. Further discussion of equation (59) will be given below.

3.2. *The searchlight problem*

Here we consider equation (16) with boundary condition (17). Converting the integro-differential equation to integral form, we find

$$\bar{\Phi}_0(s, k_1, k_2, z)=\frac{1}{\mu_0}\exp\left(-\frac{z}{\mu_0}[1+\mathrm{i}(1-\mu_0^2)^{1/2}(k_1\cos\psi_0+k_2\sin\psi_0)]\right)$$
$$+\tfrac{1}{2}c\int_0^{\infty}\mathrm{d}z'\,K(|z-z'|;k)\bar{\Phi}_0(s, k_1, k_2, z'). \tag{60}$$

The angular distribution at the surface is

$$\bar{\bar{\Phi}}(s, -\mu, \psi, k_1, k_2, 0) = \frac{c}{4\pi\mu} \int_0^\infty \mathrm{d}z'\, \bar{\bar{\Phi}}_0(s, k_1, k_2, z') \times \exp\left(-\frac{z'}{\mu}[1 + \mathrm{i}(1-\mu^2)^{1/2}(k_1 \cos\psi + k_2 \sin\psi)]\right). \tag{61}$$

In order to solve equation (60) we follow Rybicki (1971) and note that in defining the Green function $G(z, z_0)$ by

$$G(z, z_0) = \delta(z - z_0) + \tfrac{1}{2}c \int_0^\infty \mathrm{d}z'\, K(|z - z'|; k) G(z', z_0) \tag{62}$$

we may write

$$\bar{\bar{\Phi}}_0(s, k_1, k_2, z) = \frac{1}{\mu_0} \int_0^\infty \mathrm{d}z_0\, G(z, z_0) \exp\left(-\frac{z_0}{\mu_0}[1 + \mathrm{i}(1-\mu_0^2)^{1/2}(k_1 \cos\psi_0 + k_2 \sin\psi_0)]\right). \tag{63}$$

Clearly then the emergent distribution can be written as

$$\bar{\bar{\Phi}}(s, -\mu, \psi, k_1, k_2, 0) = \frac{c}{4\pi\mu\mu_0} \bar{\bar{G}}\left(\frac{1 + \mathrm{i}f}{\mu}, \frac{1 + \mathrm{i}f_0}{\mu_0} \right) \tag{64}$$

where $f = (1-\mu^2)^{1/2}(k_1 \cos\psi + k_2 \sin\psi)$ and

$$\bar{\bar{G}}(p_1, p_2) = \int_0^\infty \mathrm{d}z\, \mathrm{e}^{-p_1 z} \int_0^\infty \mathrm{d}z_0\, \mathrm{e}^{-p_2 z_0} G(z, z_0). \tag{65}$$

Thus if we can obtain the Green function $G(z, z_0)$ all other quantities of interest are available by quadrature. The surface distribution is also given from

$$\bar{\bar{\Phi}}_0(s, k_1, k_2, 0) = \frac{1}{\mu_0} \bar{\bar{G}}\left(0, \frac{1 + \mathrm{i}f_0}{\mu_0} \right), \tag{66}$$

as may be seen by integrating equation (64) and using (65) and (62).

To obtain the solution of equation (62) for $G(z, z_0)$ we introduce $M(z, z_0)$ where

$$G(z, z_0) = \delta(z - z_0) + cM(z, z_0) \tag{67}$$

which satisfies

$$M(z, z_0) = \tfrac{1}{2}c \int_0^\infty \mathrm{d}z'\, K(|z - z'|; k) M(z', z_0) + \tfrac{1}{2}K(|z - z_0|; k). \tag{68}$$

We use the technique discussed in the last section by introducing $\Psi(\eta, z, z_0)$ where

$$\left(\eta Q \frac{\partial}{\partial z} + Q^2 \right) \Psi(\eta, z, z_0) = \tfrac{1}{2}cM(z, z_0) + \tfrac{1}{2}\delta(z - z_0) \tag{69}$$

and

$$M(z, z_0) = \int_{-1}^{1} \mathrm{d}\eta\, \Psi(\eta, z, z_0) \tag{70}$$

subject to

$$\Psi(\eta, 0, z_0) = 0 \qquad \eta > 0. \tag{71}$$

By using the double Laplace transform

$$\bar{\bar{M}}(p_1, p_2) = \int_0^\infty \mathrm{d}z\, \mathrm{e}^{-p_1 z} \int_0^\infty \mathrm{d}z_0\, \mathrm{e}^{-p_2 z_0} M(z, z_0) \tag{72}$$

on equation (69) we find after some algebra

$$R(p_1, p_2) V(p_1) = 1 + c(p_1 + p_2) g(p_1, p_2) \tag{73}$$

where

$$R(p_1, p_2) = 1 + c(p_1 + p_2) \bar{\bar{M}}(p_1, p_2) \tag{74}$$

and

$$g(p_1, p_2) = \int_{-1}^{0} \frac{\mathrm{d}\eta\, \eta \bar{\Psi}(\eta, 0, p_2)}{\eta p_1 + Q}. \tag{75}$$

Using the Wiener–Hopf decomposition as discussed in § 3.1 we obtain

$$[1 + c(p_1 + p_2) \bar{\bar{M}}(p_1, p_2)] = C_0 H(1/p_1) \tag{76}$$

and

$$[1 + c(p_1 + p_2) g(p_1, p_2)] = C_0 / H(-1/p_1). \tag{77}$$

Setting $p_1 = -p_2$ gives

$$C_0 = H(1/p_2) \tag{78}$$

and since

$$(p_1 + p_2) \bar{\bar{G}}(p_1, p_2) = 1 + c(p_1 + p_2) \bar{\bar{M}}(p_1, p_2) \tag{79}$$

we find

$$\bar{\bar{G}}(p_1, p_2) = \frac{H(1/p_1) H(1/p_2)}{p_1 + p_2}. \tag{80}$$

Inversion of this function leads directly to $G(z, z_0)$. We do note, however, that

$$\begin{aligned} \bar{G}(0, p_2) &= \lim_{p_1 \to \infty} p_1 \bar{\bar{G}}(p_1, p_2) \\ &= H(1/p_2). \end{aligned} \tag{81}$$

From these results we obtain

$$\bar{\bar{\Phi}}(s, -\mu, \psi, k_1, k_2, 0) = \frac{c}{4\pi} \frac{H(\mu/(1 + \mathrm{i}f)) H(\mu_0/(1 + \mathrm{i}f_0))}{(1 + \mathrm{i}f)\mu_0 + (1 + \mathrm{i}f_0)\mu} \tag{82}$$

and

$$\bar{\bar{\Phi}}_0(s, k_1, k_2, 0) = \frac{1}{\mu_0} H\left(\frac{\mu_0}{1 + \mathrm{i}f_0}\right). \tag{83}$$

To obtain the complete solution in terms of x, y and u requires the inversion of the transforms: this is not an easy task and we defer discussion to a later paper. There seems to be no simple result as in the case of the line source.

It is worth pointing out that $M(z, 0)$ is the solution obtained by Elliott for an isotropic source on the surface of a half-space. This solution is obtained readily from equations (79) and (80), namely

$$\bar{M}(p, 0) = (H(1/p) - 1)/c. \tag{84}$$

The inversion in the p plane is straightforward and is given by Elliott (1952, 1955).

Some estimate of the collision density at the surface in the searchlight problem can be obtained by assuming that the beam is averaged over all directions, i.e. a distribution of beams all converging at the origin. Thus we must average over μ_0 and ψ_0 as follows:

$$\bar{\Phi}_0(s, k, 0) = \int_0^1 \mathrm{d}\mu_0 \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\psi_0\, H\left(\frac{\mu_0}{1 + \mathrm{i}f_0}\right). \tag{85}$$

It may be shown that this double integral reduces to a single one of the following form (see the Appendix):

$$\bar{\Phi}_0(s, k, 0) = \int_0^1 \mathrm{d}\omega\, H\left(\frac{\omega}{(1 + \omega^2 k^2)^{1/2}}\right). \tag{86}$$

Unfortunately it is not possible to reduce this integral to a simpler form as was done in the constant source case. The nearest it is possible to get to equation (86) through equation (49) is

$$\int_0^1 \frac{\mathrm{d}\omega}{1 + k^2\omega^2} H\left(\frac{\omega}{(1 + \omega^2 k^2)^{1/2}}\right) = \frac{2}{c}\left(1 - \frac{1}{H(\infty)}\right)$$
$$= \frac{2}{c}\left[1 - \left(1 - \frac{c}{k}\tan^{-1} k\right)^{1/2}\right]. \tag{87}$$

For $k = 0$ this reduces to the well known one-dimensional albedo result of

$$\Phi_0(0) = 2[1 - (1 - c)^{1/2}]/c. \tag{88}$$

A lower bound on the three-dimensional result can be obtained using the Schwarz inequality. Thus

$$\int_0^1 \frac{\mathrm{d}\omega}{1 + k^2\omega^2} H\left(\frac{\omega}{(1 + \omega^2 k^2)^{1/2}}\right) \leqslant \int_0^1 \frac{\mathrm{d}\omega}{1 + k^2\omega^2} \int_0^1 \mathrm{d}\omega\, H\left(\frac{\omega}{(1 + \omega^2 k^2)^{1/2}}\right) \tag{89}$$

and so we may write

$$\bar{\Phi}_0(s, k, 0) \geqslant \frac{2}{c}\left[1 - \left(1 - \frac{c}{k}\tan^{-1} k\right)^{1/2}\right] \frac{k}{\tan^{-1} k}. \tag{90}$$

This may be inverted in the k plane by using the technique developed in equations (55)–(62). The result can be written as

$$\Phi_0(s, x, y, 0) \geqslant \frac{4}{\pi^2 c} \int_\nu^1 \frac{\mathrm{d}t\, t^2[(c/2t)\ln[(1+t)/(1-t)] - 1]^{1/2}}{\ln[(1+t)/(1-t)]} K_0(Rt). \tag{91}$$

4. The *H* function

The generalised H function is defined in § 3.1 by equation (46) as a contour integral. It is also defined implicitly by the non-linear integral equations (48) and (49). The contour integral can be transformed to a convenient form for computation and we find

$$H\left(\frac{1}{p}\right)=\frac{p+(1+k^2)^{1/2}}{p+(\nu^2+k^2)^{1/2}}\exp\left(\frac{1}{\pi}\int_0^1\frac{\mathrm{d}\omega\,\tan^{-1}\Lambda(\omega)}{\omega(1+k^2\omega^2)^{1/2}(\omega p+(1+\omega^2k^2)^{1/2})}\right) \tag{92}$$

where

$$\Lambda(\omega)=\tfrac{1}{2}c\pi\omega(1-\tfrac{1}{2}c\omega\,\ln[(1+\omega)/(1-\omega)])^{-1}.$$

Rybicki (1971) has also shown that by extending Chandrasekhar's (1960) treatment H can be written as the infinite product

$$H_n(1/p)=\prod_{i=1}^{n}\frac{p+(\lambda_i^2+k^2)^{1/2}}{p+(\nu_i^2+k^2)^{1/2}} \tag{93}$$

where ν_i^2 are the roots of

$$1=c\sum_{i=1}^{n}\frac{\omega_i}{1-\nu^2\mu_i^2} \tag{94}$$

with $\lambda_i=1/\mu_i$. μ_i and ω_i are the Gauss–Legendre quadrature weights and nodes. As $n\to\infty$, $H_n(1/p)$ tends to the exact value. Because of the analytical difficulties involved in inverting the k transforms, we shall obtain some approximate results with the first approximation to the H function, i.e.

$$H_1(1/p)=\frac{p+(3+k^2)^{1/2}}{p+[k^2+3(1-c)]^{1/2}}. \tag{95}$$

Solutions so obtained will show the typical form of the spatial dependence of the albedo and the energy deposition.

5. Solutions using the approximate *H* function

5.1. *Line source problem*

There is a number of useful quantities that can be calculated for this problem. (1) The energy deposition at the surface

$$W(x,y,0)=\frac{E_0S_0(1-\alpha)}{2c}\frac{1}{2\pi}\int_0^\infty \mathrm{d}k\,kJ_0(kR)(H(\infty)-1) \tag{96}$$

which has already been obtained exactly in equation (59). (2) The complete energy deposition function

$$W(x,y,z)=\frac{E_0S_0(1-\alpha)}{2c}\frac{1}{2\pi}\int_0^\infty \mathrm{d}k\,kJ_0(kR)\frac{1}{2\pi\mathrm{i}}\int_c \mathrm{d}p\,\mathrm{e}^{pz}(H(\infty)H(1/p)-1)/p. \tag{97}$$

(3) The emergent angular distribution

$$\Phi(-\mu,\psi,x,y,0)=\frac{S_0}{4\pi}\frac{1}{2\pi}\int_0^\infty \mathrm{d}k\,kJ_0(kR)\frac{H(\infty)}{1+\mathrm{i}f}H\left(\frac{\mu}{1+\mathrm{i}f}\right). \tag{98}$$

(4) The emergent current, which for unit incident current is also the albedo,

$$J(x,y,0)=\frac{S_0}{4\pi}\int_0^\infty \mathrm{d}k\,kJ_0(kR)H(\infty)\int_0^1\frac{\mathrm{d}\omega\,\omega}{(1+k^2\omega^2)^{1/2}}H\left(\frac{\omega}{(1+k^2\omega^2)^{1/2}}\right). \tag{99}$$

Let us consider first the energy deposition at the surface in the H_1 approximation. Inserting equation (95) into equation (96) leads to

$$W(x, y, 0)=\frac{E_0S_0(1-\alpha)}{2c}\frac{1}{2\pi}\int_0^{\infty}\mathrm{d}k\, kJ_0(kR)\left[\left(\frac{3+k^2}{3(1-c)+k^2}\right)^{1/2}-1\right]. \tag{100}$$

By casting the quantity in square brackets in the integrand of equation (100) into the form of a Fourier integral and deforming the contour we can write

$$\left[\left(\frac{3+k^2}{3(1-c)+k^2}\right)^{1/2}-1\right]=\frac{2}{\pi}\int_{[3(1-c)]^{1/2}}^{\sqrt{3}}\mathrm{d}t\, t\left(\frac{3-t^2}{t^2-3(1-c)}\right)^{1/2}\frac{1}{t^2+k^2} \tag{101}$$

and hence

$$W(x, y, 0)=\frac{E_0S_0(1-\alpha)}{2\pi^2c}\int_{[3(1-c)]^{1/2}}^{\sqrt{3}}\mathrm{d}t\, t\left(\frac{3-t^2}{t^2-3(1-c)}\right)^{1/2}K_0(tR) \tag{102}$$

For $c=1$ and large R, equation (102) reduces to

$$W(x, y, 0)\sim\sqrt{3}E_0S_0(1-\alpha)/4\pi R, \tag{103}$$

which also agrees with the exact result from equation (59).

The emergent current $J(x, y, 0)$ is given by

$$J(x, y, 0)=\frac{S_0}{4\pi}\int_0^1\mathrm{d}\omega\,\omega\int_0^{\infty}\mathrm{d}k\, kJ_0(kR)\left(\frac{3+k^2}{3(1-c)+k^2}\right)^{1/2}\frac{1}{(1+\omega^2k^2)^{1/2}}$$
$$\times\frac{(1+\omega^2k^2)^{1/2}+\omega(3+k^2)^{1/2}}{(1+\omega^2k^2)^{1/2}+\omega[3(1-c)+k^2]^{1/2}}. \tag{104}$$

Now it is possible to write the integrand in the form of a Fourier integral and hence express the integral in terms of $K_0(x)$ instead of $J_0(x)$. However, the complexity of the result adds nothing essential to our knowledge that equation (102) does not already tell us. We shall therefore omit these calculations and pass on to a study of the searchlight problem.

5.2. Searchlight problem

As in the line source problem, quantities of interest are as follows. (1) The surface collision density from equation (63),

$$\Phi(x, y, 0)=\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{-\infty}^{\infty}\mathrm{d}k_2\exp(\mathrm{i}k_1x+\mathrm{i}k_2y)\int_0^{\infty}\mathrm{d}z_0\, G(0, z_0)$$
$$\times\exp\left(-\frac{z_0}{\mu_0}[1+\mathrm{i}(1-\mu_0^2)^{1/2}(k_1\cos\psi_0+k_2\sin\psi_0)]\right) \tag{105}$$

where

$$G(0, z_0)=\frac{1}{2\pi\mathrm{i}}\int_c\mathrm{d}p\, \mathrm{e}^{pz_0}H(1/p). \tag{106}$$

Hence we may write, after changing variables (k_1, k_2) to (k, θ),

$$\Phi(x, y, 0)=\frac{1}{2\pi\mu_0}\int_0^{\infty}\mathrm{d}z_0\,\mathrm{e}^{-z_0/\mu_0}\int_0^{\infty}\mathrm{d}k\, kJ_0(kR(z_0))G(0, z_0)$$

where

$$R^2(z_0)=\left(x-\frac{z_0}{\mu_0}(1-\mu_0^2)^{1/2}\cos\psi_0\right)^2+\left(y-\frac{z_0}{\mu_0}(1-\mu_0^2)^{1/2}\sin\psi_0\right)^2.$$

(2) The emergent surface current $J(x, y, 0)$ from equation (82) can also be written as

$$J(x,y,0)=\frac{c}{2}\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{-\infty}^{\infty}\mathrm{d}k_2\exp(\mathrm{i}k_1x+\mathrm{i}k_2y)H\left(\frac{\mu_0}{1+\mathrm{i}f_0}\right)$$
$$\times\int_0^1\frac{\mathrm{d}\omega\,\omega H[\omega(1+\omega^2k^2)^{-1/2}]}{\omega(1+\mathrm{i}f_0)+\mu_0(1+\omega^2k^2)^{1/2}} \tag{107}$$

or, more conveniently through equations (61) and (63), as

$$J(x,y,0)=\frac{c}{2\mu_0}\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{-\infty}^{\infty}\mathrm{d}k_2\exp(\mathrm{i}k_1x+\mathrm{i}k_2y)\int_0^{\infty}\mathrm{d}z\int_0^{\infty}\mathrm{d}z_0$$
$$\times\exp\left(-\frac{z}{\omega}(1+\omega^2k^2)^{1/2}\right)G(z,z_0)$$
$$\times\exp\left(-\frac{z_0}{\mu_0}[1+\mathrm{i}(1-\mu_0^2)^{1/2}(k_1\cos\psi_0+k_2\sin\psi_0)]\right) \tag{108}$$

where

$$G(z,z_0)=\left(\frac{1}{2\pi\mathrm{i}}\right)^2\int_{c_1}\mathrm{d}p_1\int_{c_2}\mathrm{d}p_2\exp(p_1z+p_2z_0)\frac{H(1/p_1)H(1/p_2)}{p_1+p_2}. \tag{109}$$

Now the expression for $\Phi(x, y, 0)$ can be recast into a more convenient form if we note that, using the first approximation to Chandrasekhar's H function, we get

$$G(0,z_0)=\delta(z_0)+\{(3+k^2)^{1/2}-[3(1-c)+k^2]^{1/2}\}\exp\{-[3(1-c)+k^2]^{1/2}z_0\} \tag{110}$$

and hence

$$\Phi(x,y,0)=\frac{\delta(x)\delta(y)}{\mu_0}+\frac{1}{2\pi\mu_0}\int_0^{\infty}\mathrm{d}z_0\,\mathrm{e}^{-z_0/\mu_0}\int_0^{\infty}\mathrm{d}k\,kJ_0(kR(z_0))$$
$$\times\{(3+k^2)^{1/2}-[3(1-c)+k^2]^{1/2}\}\exp\{-[3(1-c)+k^2]^{1/2}z_0\} \tag{111}$$

where we note that

$$\delta(x)\delta(y)=\frac{1}{2\pi}\int_0^{\infty}\mathrm{d}k\,kJ_0(k(x^2+y^2)^{1/2}). \tag{112}$$

But we may represent the quantity in the integrand of equation (111) as follows:

$$\{(3+k^2)^{1/2}-[3(1-c)+k^2]^{1/2}\}\exp\{-[3(1-c)+k^2]^{1/2}z_0\}$$
$$=\frac{2}{\pi}\int_{[3(1-c)]^{1/2}}^{\sqrt{3}}\frac{\mathrm{d}t\,t}{t^2+k^2}\{(3-t^2)^{1/2}\sin\{z_0[t^2-3(1-c)]^{1/2}\}$$
$$+[t^2-3(1-c)]^{1/2}\cos\{z_0[t^2-3(1-c)]^{1/2}\}\}$$
$$+\frac{2}{\pi}\int_{\sqrt{3}}^{\infty}\frac{\mathrm{d}t\,t}{t^2+k^2}\{[t^2-3(1-c)]^{1/2}-(t^2-3)^{1/2}\}\cos\{z_0[t^2-3(1-c)]^{1/2}\}. \tag{113}$$

Integrating over k leads to

$$\Phi(x, y, 0) = \frac{\delta(x)\delta(y)}{\mu_0} + \frac{1}{\pi^2 \mu_0} \int_0^\infty \mathrm{d}z_0\, \mathrm{e}^{-z_0/\mu_0} \bigg(\int_{[3(1-c)]^{1/2}}^{\sqrt{3}} \mathrm{d}t\, t K_0(tR(z_0))$$
$$\times \{(3-t^2)^{1/2} \sin\{z_0[t^2-3(1-c)]^{1/2}\} + [t^2-3(1-c)]^{1/2}$$
$$\times \cos\{z_0[t^2-3(1-c)]^{1/2}\}\} + \int_{\sqrt{3}}^\infty \mathrm{d}t\, t K_0(tR(z_0))$$
$$\times \{[t^2-3(1-c)]^{1/2} - (t^2-3)^{1/2}\} \cos\{z_0[t^2-3(1-c)]^{1/2}\}\bigg). \tag{114}$$

The integration over z_0 cannot be carried out analytically and further simplification of equation (114) is not possible. The case of normal incidence, when $\mu_0 = 1$, leads to a much simpler result which can be expressed as

$$\Phi(x, y, 0) = \delta(x)\delta(y) + \frac{1}{\pi^2} \int_{[3(1-c)]^{1/2}}^{\sqrt{3}} \mathrm{d}t\, t K_0[(x^2+y^2)^{1/2} t] \frac{[t^2-3(1-c)]^{1/2}[(3-t^2)^{1/2}+1]}{1+t^2-3(1-c)}$$
$$+ \frac{1}{\pi^2} \int_{\sqrt{3}}^\infty \mathrm{d}t\, t K_0[(x^2+y^2)^{1/2} t] \frac{\{[t^2-3(1-c)]^{1/2} - (t^2-3)^{1/2}\}}{1+t^2-3(1-c)}. \tag{115}$$

The limiting case of $c = 1$ and large $(x^2+y^2)^{1/2}$ becomes

$$\Phi(x, y, 0) \sim (\sqrt{3}+1)/2\pi R^3 = 0.870/\pi R^3, \tag{116}$$

which is to be compared with the inequality (91), namely

$$\Phi(x, y, 0) \geqslant 0.577/\pi R^3. \tag{117}$$

Hence the H_1 approximation obeys the exact inequality condition.

The emergent current $J(x, y, 0)$ as given by equation (108) can be rearranged to the following form:

$$J(x, y, 0) = \frac{c}{4\pi\mu_0} \int_0^\infty \mathrm{d}z_0 \int_0^\infty \mathrm{d}k\, k J_0(kR(z_0)) \int_0^1 \frac{\mathrm{d}\omega\, [(1+\omega^2 k^2)^{1/2} + \omega(3+k^2)^{1/2}]}{1-3(1-c)\omega^2}$$
$$\times (\omega\{(3+k^2)^{1/2} - [3(1-c)+k^2]^{1/2}\} \exp\{-[3(1-c)+k^2]^{1/2} z_0\}$$
$$+ [(1+\omega^2 k^2)^{1/2} - \omega(3+k^2)^{1/2}] \exp[-(1+\omega^2 k^2)^{1/2} z_0/\omega]). \tag{118}$$

To proceed, it is necessary to represent the quantity in round brackets in equation (118) as a Fourier integral. This can be done but, as in equation (104), the result does not warrant the effort in this preliminary discussion of three-dimensional effects.

6. Numerical work and discussion

The results in the previous section require rather lengthy numerical evaluation. It is not our intention to give a comprehensive survey in this respect but it will be useful to evaluate some typical surface distributions to illustrate the general trend of results. To do this we choose the energy deposition at the surface in the case of the line source and the surface collision density in the case of the searchlight problem. It is also helpful to assess the accuracy of the approximate H functions defined in § 4. This may be done rather easily through the line source problem for which a relatively simple exact solution for the surface energy deposition is available (see equation (59)). Table 1

Table 1.

	$c=1$		$c=0.9$		$c=0.5$		$c=0.2$	
R	Exact	Approx.	Exact	Approx.	Exact	Approx.	Exact	Approx.
0.1	5.276	7.197	3.125	5.374	0.375	2.485	1.149 (−3)	0.920
0.2	4.246	5.599	2.371	3.955	0.272	1.712	8.301 (−4)	0.615
0.3	3.652	4.689	1.938	3.152	0.214	1.286	6.500 (−4)	0.449
0.4	3.236	4.062	1.639	2.605	0.174	1.004	5.278 (−4)	0.341
0.5	2.919	3.592	1.414	2.200	0.145	0.801	4.378 (−4)	0.265
0.6	2.666	3.222	1.236	1.885	0.122	0.649	3.682 (−4)	0.210
0.7	2.455	2.921	1.090	1.632	0.104	0.532	3.128 (−4)	0.167
0.8	2.276	2.670	0.968	1.424	0.0896	0.439	2.678 (−4)	0.135
0.9	2.123	2.453	0.865	1.251	0.0774	0.365	2.305 (−4)	0.109
1.0	1.989	2.274	0.776	1.105	0.0672	0.304	1.994 (−4)	0.0886
1.5	1.504	1.644	0.473	0.628	0.0347	0.130	1.013 (−4)	0.0332
2.0	1.202	1.275	0.303	0.380	0.0186	0.0585	5.395 (−5)	0.0131
2.5	0.996	1.036	0.200	0.239	0.0104	0.0272	2.953 (−5)	5.328 (−3)
3.0	0.847	0.869	0.134	0.154	5.902 (−3)	0.0129	1.646 (−5)	2.206 (−3)
3.5	0.733	0.747	0.0918	0.102	3.382 (−3)	6.213 (−3)	9.284 (−6)	9.255 (−4)
4.0	0.646	0.654	0.0634	0.0685	1.956 (−3)	3.030 (−3)	5.287 (−6)	3.923 (−4)
4.5	0.575	0.581	0.0442	0.0466	1.139 (−3)	1.492 (−3)	3.032 (−6)	1.675 (−4)
5.0	0.518	0.522	0.0311	0.0321	6.672 (−4)	7.406 (−4)	1.749 (−6)	7.201 (−5)
5.5	0.472	0.474	0.0220	0.0222	3.926 (−4)	3.701 (−4)	1.013 (−6)	3.111 (−5)

shows the values of the integral terms in equations (59) and (102) for a range of values of R and c. From this table we may conclude that the first-order approximate H function becomes grossly inaccurate for $c<0.9$ and for small R (<1). Indeed, it is not difficult to show that as $c \to 0$

$$I(\text{equation (59)}) \sim \sqrt{2}\exp(-1/c)K_0(R)$$

and

$$I(\text{equation (102)}) \sim \tfrac{3}{4}\pi c K_0(\sqrt{3}R),$$

thereby demonstrating the error very clearly. However, for c near unity the H_1 function provides a reasonable description at distances greater than a mean free path from the source. Such behaviour is, of course, typical of diffusion theory and, indeed, it is possible to demonstrate that $H_1(Z)$ has diffusion-like properties, although it is rather more accurate for $Z<1$ than simple diffusion theory. The property that makes $H_1(Z)$ poor, for problems involving Fourier spatial inversion, is the large-Z behaviour. It is therefore worthwhile seeking better approximations to the H function for large Z or, for what is the same thing, $H(1/p)$ for small p. We shall not pursue this aspect of the problem here, but refer the reader to Abu-Shumays (1966, 1967), who considers various ways of obtaining accurate but simple representations of $H(Z)$.

Finally, we examine the searchlight problem. The numerical work associated with this case is rather large and we shall defer a full survey to a future report. However, as a measure of the influence of incident angular directions of the beam, we compute the mean distance of travel in the x direction as a function of incident beam direction. Thus

$$\bar{x}(\mu_0, \psi_0) = \frac{\int_{-\infty}^{\infty} \mathrm{d}x\, x \int_{-\infty}^{\infty} \mathrm{d}y\, \Phi(x, y, 0)}{\int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y\, \Phi(x, y, 0)}, \tag{119}$$

which in terms of the Fourier transform can be written

$$\bar{x}(\mu_0, \psi_0) = \frac{\mathrm{i}}{\bar{\Phi}(0,0,0)} \frac{\partial}{\partial k_1} \bar{\Phi}(k_1, k_2, 0)\bigg|_{k_1=k_2=0}. \tag{120}$$

From equation (63) we therefore obtain

$$\bar{x}(\mu_0, \psi_0) = \frac{(1-\mu_0^2)^{1/2}}{\mu_0} \cos\psi_0 \frac{\int_0^\infty \mathrm{d}z_0\, z_0\, \mathrm{e}^{-z_0/\mu_0} G_0(0, z_0)}{\int_0^\infty \mathrm{d}z_0\, G_0(0, z_0)} \tag{121}$$

where the subscript zero on G indicates that $k=0$. But because

$$\bar{G}_0(0, p) = H_0(1/p) = \int_0^\infty \mathrm{d}z_0\, \mathrm{e}^{-pz_0} G_0(0, z_0)$$

$$\bar{x}(\mu_0, \psi_0) = \frac{(1-\mu_0^2)^{1/2}}{\mu_0} \cos\psi_0 \frac{-H_0'(1/p)}{H_0(1/p)}\bigg|_{p=1/\mu_0} \tag{122}$$

which, after using the H-function equations, leads to

$$\bar{x}(\mu_0, \psi_0) = (1-\mu_0^2)^{1/2} \cos\psi_0 \tfrac{1}{2} c\mu_0 H_0(\mu_0) \int_0^1 \frac{\mathrm{d}\mu\, \mu H_0(\mu)}{(\mu+\mu_0)^2}. \tag{123}$$

The mean distance in the y direction follows if $\cos\psi_0$ above is replaced by $\sin\psi_0$.

We have evaluated the integral above using Abu-Shumays' $H_{11}^{(1)}$ approximation and our results are given in table 2. For the sake of example, we give only $\bar{x}(\mu_0, 0)$, i.e. with the incident beam in the x–z plane. The results show an interesting behaviour. For normal incidence the value of $\bar{x}$ is zero, a fact which follows from symmetry. As the

Table 2.

μ_0 \ c	1	0.9	0.5
0	0	0	0
0.01	0.0285	0.0216	0.0101
0.05	0.102	0.0715	0.0309
0.1	0.167	0.111	0.0456
0.2	0.261	0.160	0.0607
0.3	0.325	0.187	0.0666
0.4	0.367	0.199	0.0677
0.5	0.390	0.202	0.0655
0.6	0.394	0.195	0.0610
0.7	0.379	0.180	0.0543
0.8	0.338	0.156	0.0452
0.9	0.259	0.115	0.0324
0.95	0.190	0.0831	0.0230
0.97	0.149	0.0649	0.0179
0.99	0.0872	0.0378	0.0103
1	0	0	0

beam moves away from the normal directions the mean distance of travel increases, passes through a maximum value of about 0.5 mean free paths at $\mu_0 \simeq 0.5$ (60°) and decreases to zero at grazing incidence. The latter behaviour is expected because at grazing incidence most of the particles that would normally travel long distances have leaked out and do not contribute to the weighting. Similar calculations may be performed for the second and higher moments.

The main value of the above investigation is to throw light on the complex interaction between source direction and the spatial distribution of particles. For realistic scattering models and boundary conditions, resort is usually made to infinite medium spatial moment techniques. The present work provides an exact solution against which such approximations may be tested for accuracy and convergence. Although the present method can be extended to cascade problems, it is unlikely that realistic scattering models can be dealt with because of the restriction to an energy-independent mean free path.

Appendix

We wish to prove the identity

$$\frac{1}{2\pi}\int_0^1 \mathrm{d}\mu_0 \int_0^{2\pi} \mathrm{d}\psi_0\, H\left(\frac{\mu_0}{1+\mathrm{i}f_0}\right) \equiv \int_0^1 \mathrm{d}\omega\, H\left(\frac{\omega}{(1+\omega^2k^2)^{1/2}}\right) \tag{A.1}$$

where

$$f_0 = (1-\mu_0^2)^{1/2}(k_1 \cos\psi_0 + k_2 \sin\psi_0) \qquad \text{and} \qquad k^2 = k_1^2 + k_2^2.$$

The left-hand side of equation (A.1) can be written as a contour integral where $H(-1/p)$ is analytic in C:

$$\text{LHS} = \frac{1}{2\pi \mathrm{i}}\int_C \mathrm{d}p\, H\left(-\frac{1}{p}\right)\frac{1}{2\pi}\int_0^1 \mathrm{d}\mu_0 \int_0^{2\pi} \mathrm{d}\psi_0 \frac{1}{p+(1+\mathrm{i}f_0)/\mu_0} \tag{A.2}$$

$$= \frac{1}{2\pi \mathrm{i}}\int_C \mathrm{d}p\, H\left(-\frac{1}{p}\right)\frac{1}{2\pi}\int_0^1 \mathrm{d}\mu_0 \int_0^{2\pi} \mathrm{d}\psi_0 \int_0^\infty \mathrm{d}t\, \exp\{-t[p+(1+\mathrm{i}f_0)/\mu_0]\} \tag{A.3}$$

$$= \frac{1}{2\pi \mathrm{i}}\int_C \mathrm{d}p\, H\left(-\frac{1}{p}\right)\int_0^1 \mathrm{d}\mu_0\, \mathrm{e}^{-t/\mu_0}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-tp}\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\psi_0\, \mathrm{e}^{-\mathrm{i}tf_0/\mu_0} \tag{A.4}$$

$$= \frac{1}{2\pi \mathrm{i}}\int_C \mathrm{d}p\, H\left(-\frac{1}{p}\right)\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-tp}\int_0^1 \mathrm{d}\mu_0\, \mathrm{e}^{-t/\mu_0} J_0\left(\frac{kt(1-\mu_0^2)^{1/2}}{\mu_0}\right). \tag{A.5}$$

But from

$$\int_0^1 \frac{\mathrm{d}\omega}{\omega^2}\,\mathrm{e}^{-\alpha/\omega} J_0\left(\frac{\beta}{\omega}(1-\omega^2)^{1/2}\right) = \exp\frac{[-(\alpha^2+\beta^2)^{1/2}]}{(\alpha^2+\beta^2)^{1/2}} \tag{A.6}$$

we can prove that

$$\int_0^1 \mathrm{d}\mu_0\, \mathrm{e}^{-t/\mu_0} J_0\left(\frac{kt(1-\mu_0^2)^{1/2}}{\mu_0}\right) = \int_0^1 \mathrm{d}\omega\, \exp\left(-\frac{t}{\omega}(1+k^2\omega^2)^{1/2}\right). \tag{A.7}$$

Hence

$$\text{LHS} = \frac{1}{2\pi \mathrm{i}} \int_C \mathrm{d}p\, H\left(-\frac{1}{p}\right) \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-tp} \int_0^1 \mathrm{d}\omega\, \mathrm{e}^{[-t(1+k^2\omega^2)^{1/2}/\omega]} \tag{A.8}$$

$$= \frac{1}{2\pi \mathrm{i}} \int_C \mathrm{d}p\, H\left(-\frac{1}{p}\right) \int_0^1 \frac{\mathrm{d}\omega}{p + (1+k^2\omega^2)^{1/2}/\omega} \tag{A.9}$$

$$= \int_0^1 \mathrm{d}\omega\, H\left(\frac{\omega}{(1+k^2\omega^2)^{1/2}}\right). \tag{A.10}$$

References

Abu-Shumays I 1966 *Nucl. Sci. Eng.* **26** 430
—— 1967 *Nucl. Sci. Eng.* **27** 607
Case K M and Zweifel P F 1967 *Linear Transport Theory* (Reading, Mass.: Addison–Wesley) p 25
Chandrasekhar S 1960 *Radiative Transfer* (New York: Dover)
Davison B 1957 *Neutron Transport Theory* (London: Oxford University Press) p 66
Elliott J P 1952 *AERE, Harwell, Report AERE T/R 972*
—— 1955 *Proc. R. Soc.* A **228** 424
Khalafi F and Williams M M R 1980 *Radiat. Effects* **46** 175
Rybicki G B 1971 *J. Quant. Spectrosc. Radiat. Transfer* **11** 837
Williams M M R 1968 *Nukleonika* **11** 219
—— 1971 *Mathematical Methods in Particle Transport Theory* (London: Butterworth) p 152
—— 1978 *J. Phys. D: Appl. Phys.* **11** 801
—— 1979a *Prog. Nucl. Energy* **3** 1
—— 1979b *Ann. Nucl. Energy* **6** 145